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(Residential Autonomous College under University of Calcutta)

B.A./B.Sc. SECOND SEMESTER EXAMINATION, MAY 2014

FIRST YEAR

MATHEMATICS(Honours)

Paper : II

Date : 23/05/2014

Time : 11 am – 3 pm

Full Marks : 100

[Use a Separate Answer Book for each group]

Group – A

Answer **any five** questions from **Q.No. 1 to Q.No. 8** :

1. If n be a positive integer, prove that— [5]

$$\frac{1}{2\sqrt{n+1}} < \frac{1.3.5...(2n-1)}{2.4.6...2n} < \frac{1}{\sqrt{2n+1}}$$

2. State the theorem on weighted mean related to inequality. Apply this to find the maximum value of $(1-x)(1-y)(1-z)$ when x, y, z > 0 and x+y+z = 1. [2+3]

3. a) Use De Moivre's theorem to show that the roots of the equation $z^n = (z+1)^n$, where n is a positive integer > 1, representing the points in z-plane, are collinear.

   b) Find the general values and the principal value of $(-1)^{\sqrt{2}}$. [3+2]

4. Prove that $\sin\left[i\log\frac{a-ib}{a+ib}\right] = \frac{2ab}{a^2+b^2}$. [5]

5. Solve the equation : $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$, given that the sum of two of the roots is zero. [5]

6. Transform the equation $x^4 + 4x^3 + 7x^2 + 6x - 4 = 0$ into one which has no second degree term and hence solve the given equation. [5]

7. Reduce the reciprocal equation $3x^6 + x^5 - 27x^4 + 27x^2 - x - 3 = 0$ to a reciprocal equation of the standard form and solve it. [5]

8. Solve by Ferrari's method the equation : $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$. [5]

Answer **any five** questions from **Q.No. 9 to Q.No. 16** :

9. a) Let $f:[a,b] \to \mathbb{R}$ be a continuous function. Show that $S = \{x \in [a,b] : f(x) = 0\}$ is a compact subset of $[a,b]$.

   b) Exhibit an open cover of the set $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ that has no finite subcover. Is this set compact? Justify your answer. [2+3]

10. a) Test the convergence of the series : $\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \dots$.

    b) Examine the convergence of the series : $1 + \frac{2^2}{3^2} + \frac{2^2.4^2}{3^2.5^2} + \frac{2^2.4^2.6^2}{3^2.5^2.7^2} + \dots$. [2+3]

11. a) State Abel's theorem for a convergent series of positive real numbers.

    b) Prove that an absolutely convergent series is convergent. Show that the series $1 - \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} - \frac{1}{6^2} + \dots$ is convergent. [1+4]

12. Let I = $[a,b]$ be a closed and bounded interval and $f:[a,b] \to \mathbb{R}$ be continuous on I. Then prove that $f(I) = \{f(x) : x \in I\}$ is a closed and bounded interval. [5]

13. a) A function $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and $x_1, x_2, ..., x_n \in [a,b]$. Prove that there is a point $c\in[a,b]$ such that $f(c)=\dfrac{f(x_1)+f(x_2)+...+f(x_n)}{n}$.

b) Prove that the function $f(x)=\sin\dfrac{1}{x}, x\in(0,1)$ is not uniformly continuous on (0,1). [3+2]

14. a) Let $f:[0,1]\to\mathbb{R}$ be a continuous function on [0,1]. Show that $\left\{f\left(1-\dfrac{1}{n}\right)\right\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$.

b) Let I be an interval in $\mathbb{R}$ and $f:I\to\mathbb{R}$ be a function such that f' exists and is bounded on I. Prove that f is uniformly continuous on I. [2+3]

15. a) Prove that $\sin x < x$ in $0<x<\dfrac{\pi}{2}$ and $\sin x<\tan x$ in $0<x<\dfrac{\pi}{2}$.

b) Prove that the equation $(x-1)^3+(x-2)^3+(x-3)^3+(x-4)^3=0$ has only one real root. [3+2]

16. a) Prove that between any two real roots of the equation $e^x\sin x+1=0$, there is at least one real root of the equation $\tan x+1=0$.

b) Find a and b such that $\lim\limits_{x\to0}\dfrac{ae^x+be^{-x}+2\sin x}{\sin x+x\cos x}=2$. [3+2]

Group – B

Answer **any four** questions from **Q.No. 17 to Q.No. 22** :

17. a) By Laplace's expansion show that $\begin{vmatrix} a & -b & -a & b \\ b & a & -b & -a \\ c & -d & c & -d \\ d & c & d & c \end{vmatrix}=4(a^2+b^2)(c^2+d^2)$.

b) If A be a symmetric matrix of order m and P be an mxn matrix, prove that P^tAP is a symmetric matrix. [3+2]

18. Prove that the matrix $\dfrac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix}$ is orthogonal.

Utilise this to solve the equations :

$$\begin{aligned} x-2y+2z&=2 \\ 2x-y-2z&=1 . \\ 2x+2y+z&=7 \end{aligned}$$

[2+3]

19. Obtain the fully reduced normal form of the matrix $A=\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{pmatrix}$. Find the non-singular matrices P, Q such that PAQ is the fully reduced normal form. [3+2]

20. Show that the following set of vectors (1,1,0,1), (1,–2,0,1), (1,0,–1,2) is linearly independent in $\mathbb{R}^4$. Extend this set to a basis of $\mathbb{R}^4$. Express $\overline{x}=(x_1,x_2,x_3,x_4)$ in terms of the basis so formed. [5]

21. Find dim $S\cap T$, where S and T are subspaces of the vector space $\mathbb{R}^4$ given by $S=\{(x,y,z,w)\in\mathbb{R}^4 : 2x+y+3z+w=0\}$, $T=\{(x,y,z,w)\in\mathbb{R}^4 : x+2y+z+3w=0\}$. [5]

22. a) Find the coordinates of the vector $\alpha = (0,3,1)$ in $\mathbb{R}^3$ relative to the basis $(\alpha_1, \alpha_2, \alpha_3)$ where $\alpha_1 = (1,1,0)$, $\alpha_2 = (1,0,1)$, $\alpha_3 = (0,1,1)$.

b) Find a basis of the vector space $\mathbb{R}^3$ that contains the vectors (1,2,1) and (3,6,2). [3+2]

Answer **<u>any three</u>** questions from **<u>Q.No. 23 to Q.No. 27</u>** :

23. a) Verify that $x_1 = 2, x_2 = 1, x_3 = 3$ is a feasible solution to the equations :

$4x_1 + 2x_2 - 3x_3 = 1$

$6x_1 + 4x_2 - 5x_3 = 1$

Reduce this to a basic feasible solution.

b) Use simplex method to solve the following L.P.P.

Maximize $z = 5x_1 + 2x_2$

subject to $6x_1 + 10x_2 \le 30$

$10x_1 + 4x_2 \le 20;\ x_1, x_2 \ge 0$.

Is the solution unique? If not, write down the convex combination of the alternative optima. [4+6]

24. a) Find the optimal solution of the following problem by solving its dual :

Maximize $z = 3x_1 + 4x_2$

subject to $x_1 + x_2 \le 10$

$2x_1 + 3x_2 \le 18$

$x_1 \le 8$

$x_2 \le 6$

$x_1, x_2 \ge 0$.

b) Prove that every extreme point of the convex set of all feasible solutions of the system $Ax = b, x \ge 0$ corresponds to a basic feasible solution. [5+5]

25. a) Prove that the number of basic variables in a transportation problem is at most $(m + n - 1)$.

b) Prove that if a constraint of a primal problem is an equation then the corresponding dual variable is unrestricted in sign.

c) Solve the following Travelling Salesman problem : [4+2+4]

	A	B	C	D
A	∞	5	8	4
B	5	∞	7	4
C	8	7	∞	8
D	4	4	8	∞

26. a) Solve the following problem by two-phase method :

Maximize $z = 4x_1 + 7x_2$

subject to $12x_1 + 7x_2 \le 42$

$5x_1 + 4x_2 \le 20$

$2x_1 + 3x_2 \ge 6$

$x_1, x_2 \ge 0$.

b) Show that a basic feasible solution of the system of equations $Ax = b,\ x \ge 0$ corresponds to an extreme point of the convex set of feasible solutions. [6+4]

27. a) Find the optimal solution of the following transportation problem :

	D_1	D_2	D_3	D_4	a_i
O_1	19	30	50	10	7
O_2	70	30	40	60	9
O_3	40	8	70	20	18
b_j	5	8	7	14	

b) The Head of the department has five jobs A,B,C,D,E and five sub-ordinates V,W,X,Y,Z. The number of hours each man would take to perform each job is as follows :

	V	W	X	Y	Z
A	3	5	10	15	8
B	4	7	15	18	8
C	8	12	20	20	12
D	5	5	8	10	6
E	10	10	15	25	10

How would the jobs be allocated to minimize the total time. [5+5]